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DYNAMICS OF

MULTI-BODY SYSTEMS

by

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The dynamic equations of multi-body systems in the form of open chains are derived by applying the principles of linear and angular momentum to each individual member in the chain. This results in the appearance of constraint forces and torques in the dynamic equations. Using more or less classical approach these unknown forces and torques can be eliminated. Another approach is to approximate these forces by elastic and viscous forces by allowing small violations of the constraints. The well-known elimination procedure leads to a small densely coupled system of equations while the lesser-known procedure of approximating the constraint forces and torques yields a large but less densely coupled system. Both these procedures are first explained in the context of a single rigid body and then applied to a system of rigid bodies in an open chain where each body is coupled directly to at most two neighbours.

DYNAMICS OF MULTI-BODY SYSTEMS

DYNAMIQUES DES SYSTÈMES DE PLUSIEURS CORPS

by/par

I.H. Mufti

SUMMARY

The dynamic equations of multi-body systems in the form of open chains are derived by applying the principles of linear and angular momentum to each individual member in the chain. This results in the appearance of constraint forces and torques in the dynamic equations. Using more or less classical approach these unknown forces and torques can be eliminated. Another approach is to approximate these forces by elastic and viscous forces by allowing small violations of the constraints. The well-known elimination procedure leads to a small densely coupled system of equations while the lesser-known procedure of approximating the constraint forces and torques yields a large but less densely coupled system. Both these procedures are first explained in the context of a single rigid body and then applied to a system of rigid bodies in an open chain where each body is coupled directly to at most two neighbours.

RÉSUMÉ

Les équations dynamiques d'un système de plusieurs corps disposés en chaînes ouvertes sont dérivées en appliquant les principes de la quantité de mouvement et du moment cinétique à chaque membre de la chaîne. Il en résulte que des contraintes de force et de couple se dégagent des équations dynamiques. Par une démarche plus ou moins classique, on peut éliminer ces forces et ces couples inconnus. Une autre solution consiste à remplacer ces forces par des forces d'élasticité et de viscosité approximatives, en violant quelque peu les contraintes. La méthode classique d'élimination donne un petit système d'équations fortement couplées, tandis que la méthode moins répandue d'approximation des contraintes de force et de couple produit un grand système d'équations moins fortement couplées. Les deux méthodes sont appliquées d'abord à l'étude d'un corps rigide unique, puis à l'étude d'un système de corps rigides disposés en une chaîne ouverte où chaque corps est couplé directement à au plus deux corps voisins.

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CONTENTS

DYNAMICS OF MULTI-BODY SYSTEMS

1.0 INTRODUCTION

During the last two decades there has been a considerable interest in the study of multi-body systems, that is systems consisting of a finite number of interconnected rigid bodies. Typical examples are manipulators, linkages in machines and mechanism of human body provided each part is considered as rigid. Although the general principles of obtaining the equations of motion of such systems have been known since the days of Euler (1707-1783) and Lagrange (1726-1813) yet there is a need for finding general and computer oriented methods as the formalism suitable for analytical purposes may not be convenient for computer simulation. Most of the recent work (Refs. 1-3) is therefore devoted to obtaining methods that are efficient and general enough to be applicable to a wide variety of multi-body systems with a minimum amount of preparatory work.

The purpose of this report is to explain in as simple a way as possible some of the methods of deriving the equations of motion of multi-body systems. To accomplish this we assume that we are given a system of n rigid bodies connected together in an open chain such that each body is coupled directly to at most two neighbours as shown in Figure 1.
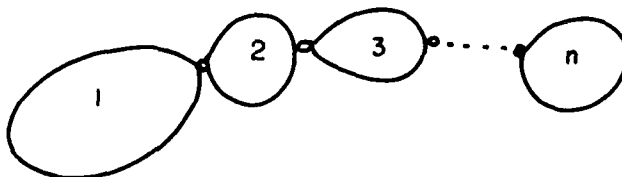
FIG. 1

The bodies are attached to each other either by ball-and-socket joint, universal joint or by pin joint*.

The principles of linear and angular momentum (Newton's and Eulers's law) applied to each individual body in the chain provide a simple way of writing the equations of motion of a multi-body system. However, this results in the appearance of unknown constraint forces and torques. There are now essentially two approaches for removing them. In the classical approach these forces and torques are eliminated from the 6n second-order differential equations resulting in at most 3n+3 equations corresponding to the number of degrees of freedom. The equations so obtained are complicated and densely coupled. In the other approach each body is kept as a free body and the constraint forces and torques are approximated by elastic and viscous forces by allowing small violations of the constraints. (They can also be approximated numerically by solving the equations of motion and constraint equations simultaneously.) The number of equations to be solved in this approach is more than that solved in the first approach. However, they are simple and less densely coupled and may be as convenient to solve on the computer as the smaller number of more complicated equations.

Apart from the introductory section the report is divided into four sections. In Section 2 we present the equations of motion of a single rigid body in Newton-Euler's form as well as in Lagrange's form and establish the connection between the two sets of equations. The motion of a single body with constraints is discussed next in Section 3. Using Lagrange multiplier method the idea of constraint forces and torques is explained. The equations of motion are formulated using the two approaches

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* Relative motion of two adjacent bodies is a pure rotation with one, two or three degrees of freedom according as it is a pin joint, universal joint or ball-and-socket joint.

mentioned above. In Section 4 the ideas developed in Sections 2 and 3 are extended to multi-body systems. The case when all connections are ball-and-socket joints is considered first. This is followed by the case when the joint may be universal or pin joint. Conclusions are presented in Section 5.

2.0 RIGID BODY EQUATIONS OF MOTION

The equations of motion of a rigid body can be written down by applying the principles of linear and angular momentum or by using the Lagrange's equation. The principles of linear and angular momentum yield

$$m \frac{d^2 r_c}{dt^2} = F \tag{1}$$

$$\frac{dL_c}{dt} = G \tag{2}$$

where r_c is the position vector of the centre of mass, m is the mass of the body, L_c is the angular momentum about the centre of mass, F is the sum of external forces and G is the sum of external torques with respect to the centre of mass C. Equations of motion (1) and (2) are valid in an inertial (spatially-fixed) frame of reference 0XYZ (see Fig. 2). Since $L_c = I\omega$, where I is the inertia matrix about the centre of mass and ω is the absolute angular velocity Equation (2) will take a simple form if the components of the angular momentum are referred to a body-fixed axes, in particular to the principal axes. (I is a constant diagonal matrix when referred to the principal axes.)
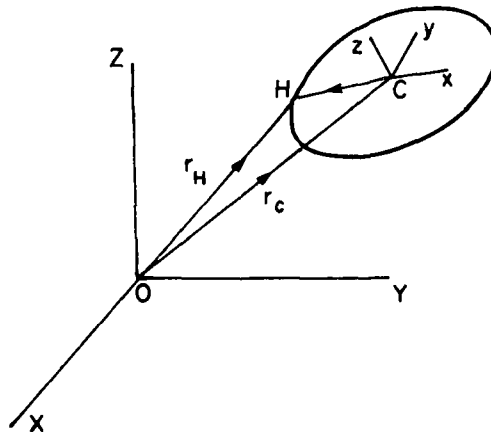
FIG. 2

The body-fixed set of axes are obtained from the inertial axes by performing three successive rotations. Before the first rotation the two set of axes are parallel. The first rotation is made about the Z-axis through an angle ψ counterclockwise to obtain an intermediate axes $C x_1 y_1 Z$. A new set of axes x_2, y_1, z_2 is now obtained by rotating the axes x_1, y_1, Z by an angle θ counterclockwise about the y_1-axis. The final rotation is carried out about the x_2-axis through an angle ϕ to obtain the body-fixed axes x,y,z. The transformation matrix R connecting the body-fixed axes to the inertial axes is given by (see, for example, Refs. 4, 5)

$$R = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \qquad (3)$$

where

$$R_{11} = \cos\psi \, \cos\theta, \; R_{12} = \cos\psi \, \sin\theta \, \sin\phi - \sin\psi \, \cos\phi$$

$$R_{13} = \cos\psi \, \sin\theta \, \cos\phi + \sin\psi \, \sin\phi, \; R_{21} = \sin\psi \, \cos\theta$$

$$R_{22} = \sin\psi \, \sin\theta \, \sin\phi + \cos\psi \, \cos\phi, \; R_{23} = \sin\psi \, \sin\theta \, \cos\phi - \cos\psi \, \sin\phi$$

$$R_{31} = -\sin\theta, \; R_{32} = \cos\theta \, \sin\phi, \; R_{33} = \cos\theta \, \cos\phi.$$

The three angles ψ, θ, ϕ specifying the orientation of the body are called Euler's angles. In terms of the Euler angle rates $\dot{\phi}, \dot{\theta}, \dot{\psi}$ (dot denotes derivatives with respect to time) the components $\omega_x, \omega_y, \omega_z$ of the angular velocity ω expressed in body-fixed axes are given by

$$\omega_x = \dot{\phi} - \dot{\psi} \, \sin\theta$$

$$\omega_y = \dot{\psi} \, \cos\theta \, \sin\phi + \dot{\theta} \, \cos\phi \qquad (4)$$

$$\omega_z = \dot{\psi} \, \cos\theta \, \cos\phi - \dot{\theta} \, \sin\phi.$$

Let the body-fixed axes be in the direction of principal axes of inertia and let I_x, I_y, I_z denote the moments of inertia about three axes. Remembering that the inertial derivative $\frac{d}{dt}(I\omega) = I\dot{\omega} + \omega \times I\omega$ in the body-fixed axes, Equation (2) becomes

$$I_x \, \dot{\omega}_x - (I_y - I_z)\omega_y \omega_z = G_x$$

$$I_y \, \dot{\omega}_y - (I_z - I_x)\omega_x \omega_z = G_y \qquad (5)$$

$$I_z \, \dot{\omega}_z - (I_x - I_y)\omega_x \omega_y = G_z.$$

These are Euler's equations of motion for a single rigid body. Euler's Equation (5) together with Equation (4) and Newton's Equation (1) yield six second-order differential equations for the determination of $x_c, y_c, z_c, \phi, \theta$ and ψ.

We mention that from computational point of view it may be convenient to solve Equation (5) and Equation (4) when inverted as first-order equations in $\omega_x, \omega_y, \omega_z, \phi, \theta$ and ψ. Solving Equation (4) for $\dot{\phi}, \dot{\theta}, \dot{\psi}$ we obtain

$$\dot{\phi} = \omega_x + \tan\theta \, (\omega_y \, \sin\phi + \omega_z \, \cos\phi)$$

$$\dot{\theta} = \omega_y \, \cos\phi - \omega_z \, \sin\phi \qquad (6)$$

$$\dot{\psi} = \frac{1}{\cos\theta} \, (\omega_y \, \sin\phi + \omega_z \, \cos\phi)$$

for $\theta \neq \pi/2$.

Let us now determine the equations of motion using Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i; \quad i=1, 2, \ldots, 6, \tag{7}$$

where T denotes the kinetic energy, q_i the ith generalized co-ordinate, $\dot{q}_i$ the generalized velocity, and Q_i the ith generalized force. Assuming again the body-fixed axes in the direction of the principal axes we have

$$T = \frac{1}{2} m \left(\dot{x}_c^2 + \dot{y}_c^2 + \dot{z}_c^2\right) + \frac{1}{2}\left(I_x \omega_x^2 + I_y \omega_y^2 + I_z \omega_z^2\right) \tag{8}$$

with $x_c, y_c, z_c, \phi, \theta, \psi$ as the six generalized co-ordinates. Substituting Equation (8) into Equation (7) we obtain the Newton's Equation (1) for the centre of mass from the first three equations with x_c, y_c, z_c as generalized co-ordinates:

$$m \frac{d^2 x_c}{dt^2} = F_1, \quad m \frac{d^2 y_c}{dt^2} = F_2, \quad m \frac{d^2 z_c}{dt^2} = F_3 \tag{9}$$

where F_1, F_2, F_3 are the components of F expressed in inertial frame. For the other three co-ordinates ϕ, θ, ψ we get

$$p_\phi \equiv \frac{\partial T}{\partial \dot{\phi}} = I_x \omega_x, \quad \frac{\partial T}{\partial \phi} = (I_y - I_z)\omega_y \omega_z$$

$$p_\theta \equiv \frac{\partial T}{\partial \dot{\theta}} = I_y \omega_y \cos\phi - I_z \omega_z \sin\phi, \quad \frac{\partial T}{\partial \theta} = -I_x \omega_x \dot{\psi} \cos\theta - I_y \omega_y \dot{\psi} \sin\theta \sin\phi - I_z \omega_z \dot{\psi} \sin\theta \cos\phi$$

$$\tag{10}$$

$$p_\psi \equiv \frac{\partial T}{\partial \dot{\psi}} = -I_x \omega_x \sin\theta + I_y \omega_y \cos\theta \sin\phi + I_z \omega_z \cos\theta \cos\phi, \quad \frac{\partial T}{\partial \psi} = 0$$

where p_ϕ, p_θ and p_ψ denote the generalized momenta. Thus the Lagrange's equation corresponding to the co-ordinates ϕ, θ, ψ can be written as

$$\dot{p}_\phi - (I_y - I_z)\omega_y \omega_z = G_1$$

$$\dot{p}_\theta + (\omega_y \sin\phi + \omega_z \cos\phi)(I_x \omega_x + \tan\theta \sin\phi I_y \omega_y + \tan\theta \cos\phi I_z \omega_z) = G_2 \tag{11}$$

$$\dot{p}_\psi = G_3$$

since

$$I_x \omega_x \dot{\psi} \cos\theta + (I_y \omega_y \sin\phi + I_z \omega_z \cos\phi) \dot{\psi} \sin\theta$$

$$= I_x \omega_x (\omega_y \sin\phi + \omega_z \cos\phi) + (I_y \omega_y \sin\phi + I_z \omega_z \cos\phi)(\omega_y \sin\phi + \omega_z \cos\phi)\tan\theta$$

from Equation (6). Equations defining p_ϕ, p_θ, p_ψ can be inverted to give

$$\omega_x = p_\phi / I_x$$

$$\omega_y = \frac{1}{I_y} \left[\frac{\sin\phi}{\cos\theta} (p_\psi + p_\phi \sin\theta) + p_\theta \cos\phi \right] \tag{12}$$

$$\omega_z = \frac{1}{I_z} \left[\frac{\cos\phi}{\cos\theta} (p_\psi + p_\phi \sin\theta) - p_\theta \sin\phi \right].$$

For rotational equations, Equations (11) and (6) with ω_x, ω_y, ω_z defined by Equation (12) may therefore be used instead of Euler's Equation (5) and Equation (6). By comparing Equations (5) and (11) it is easily seen (note the expressions for p_ϕ, p_θ, p_ψ) that

$$G_1 = G_x$$

$$G_2 = G_y \cos\phi - G_z \sin\phi \tag{13}$$

$$G_3 = -G_x \sin\theta + \cos\theta (G_y \sin\phi + G_z \cos\phi).$$

By inverting Equation (13) we get G_x, G_y, G_z in terms of G_1, G_2, G_3 :

$$G_x = G_1$$

$$G_y = G_2 \cos\phi + \frac{\sin\phi}{\cos\theta} (G_1 \sin\theta + G_3) \tag{14}$$

$$G_z = -G_2 \sin\phi + \frac{\cos\phi}{\cos\theta} (G_1 \sin\theta + G_3).$$

3.0 RIGID BODY WITH CONSTRAINTS

Before considering the multi-body dynamics let us first consider the case of a single body with constraints. The constraints reduce the number of degrees of freedom. For example, if a point of the body is fixed the motion is a pure rotation, called gyroscopic motion, and has three degrees of freedom. For a pin joint we have only one degree of freedom and so on for other types of joints.

3.1 Gyroscopic Motion

Let us assume that a point, say H, of the body is fixed. Let CH = c be expressed in body-fixed axes. From Figure 2 we can write the constraint relation as:

$$r_H = r_c + Rc = \alpha \tag{15}$$

where α is a constant vector. Equation (15) is a vector equation and is equivalent to three scalar equations. Due to constraint Equation (15) there is a constraint force f_H acting on the body at the hinge point H. The Equations of motion (1) and (2) must be replaced by

$$m \frac{d^2 r_c}{dt^2} = F + f_H \tag{16}$$

$$I\dot{\omega} + \omega \times I\omega = G + c \times R^T f_H \tag{17}$$

where it is assumed that f_H is expressed in the inertial frame and the superscript T denotes the transpose of a matrix. Since constraint force f_H is unknown, Equations (16) and (17) cannot be solved.

There are now two ways to proceed. Either eliminate the constraint force f_H and reduce the number of equations to three or approximate the unknown force f_H.

The usual procedure to eliminate the constraint force f_H is to differentiate twice the constraint Equation (15):

$$\frac{d^2 r_H}{dt^2} = \frac{d^2 r_c}{dt^2} + R(\dot{\omega} \times c + \omega \times (\omega \times c)) = 0. \tag{18}$$

Eliminating $\dfrac{d^2 r_c}{dt^2}$ from Equations (16) and (18) we obtain

$$f_H = -mR(\dot{\omega} \times c + \omega \times (\omega \times c)) - F. \tag{19}$$

Substituting Equation (19) into the rotational Equation (17) we obtain

$$I\dot{\omega} + \omega \times I\omega = G - mc \times (\dot{\omega} \times c + \omega \times (\omega \times c)) - c \times R^T F \tag{20}$$

where we have used the fact that $R^{-1} = R^T$. Since

$$c \times (\dot{\omega} \times c + \omega \times (\omega \times c)) = ((c \cdot c)E - cc^T)\dot{\omega} + \omega \times ((c \cdot c)E - cc^T)\omega,$$

Equation (20) can be rewritten as

$$I_H \dot{\omega} + \omega \times I_H \omega = G - c \times R^T F \tag{21}$$

where I_H, the inertia matrix about fixed point H, is given by

$$I_H = I + m ((c \cdot c)E - cc^T),$$

$(c \cdot c)$ is the dot product $c^T c$ and E is the unit 3×3 matrix. Equation (21) is the desired rotational equation representing the three degrees of freedom which can be solved to determine the orientation of the body. Having solved this, the constraint force f_H can be determined from Equation (19).

We mention that Equation (21) can be obtained directly by applying the principle of angular momentum about the point H. However, we would not obtain the hinge force that may be required to monitor the stress on the joint.

In writing down the modified Equations of motion (16) and (17) it was assumed that the constraint Equation (15) gives arise to the constraint force f_H. This can also be obtained by using the Lagrange multiplier method (Refs. 5, 6). The kinetic energy T for the Lagrange method is augmented by the introduction of Lagrange multipliers λ_1, λ_2, λ_3:

$$L = T + \lambda_1 (x_c + (Rc)_1 - \alpha_1) + \lambda_2 (y_c + (Rc)_2 - \alpha_2) + \lambda_3 (z_c + (Rc)_3 - \alpha_3) \tag{22}$$

where T is defined as in Equation (8) and $(Rc)_i$, α_i, $i = 1, 2, 3$ denote the components of the vectors Rc, α in the inertial frame. The equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_c}\right) - \frac{\partial L}{\partial x_c} = Q_1 \qquad \text{yields} \qquad m \frac{d^2 x_c}{dt^2} = F_1 + \lambda_1.$$

Therefore λ_1 can be identified with the first component of f_H. Similarly λ_2 and λ_3 can be equated to the second and the third components of f_H respectively. To see that the torque due to constraint force f_H is given by $c \times R^T f_H$ we compute the contribution of augmented terms to the Lagrange equations. Denoting these terms by g_{H1}, g_{H2}, g_{H3} we have

$$g_{H1} = \lambda^T \frac{\partial}{\partial \phi} (Rc), \quad g_{H2} = \lambda^T \frac{\partial}{\partial \theta} (Rc), \quad g_{H3} = \lambda^T \frac{\partial}{\partial \psi} (Rc)$$

where $\lambda^T = (\lambda_1 \ \lambda_2 \ \lambda_3)$. Since $\dot{R} = R \, \tilde{\omega}$ (Ref. 4), where

$$\tilde{\omega} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

it follows that

$$\frac{\partial R}{\partial \phi} = R \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\frac{\partial R}{\partial \theta} = R \begin{pmatrix} 0 & \sin\phi & \cos\phi \\ -\sin\phi & 0 & 0 \\ -\cos\phi & 0 & 0 \end{pmatrix}$$

$$\frac{\partial R}{\partial \psi} = R \begin{pmatrix} 0 & -\cos\theta\cos\phi & \cos\theta\sin\phi \\ \cos\theta\cos\phi & 0 & \sin\theta \\ -\cos\theta\sin\phi & -\sin\theta & 0 \end{pmatrix}$$

Let g_{Hx}, g_{Hy}, g_{Hz} denote the components of $g_H = c \times R^T f_H = \tilde{c} \, R^T f_H$. Then

$$g_{Hx} = (0 \qquad -c_3 \qquad c_2) R^T f_H$$

$$g_{Hy} = (c_3 \qquad 0 \qquad -c_1) R^T f_H$$

$$g_{Hz} = (-c_2 \qquad c_1 \qquad 0) R^T f_H.$$

Evaluating g_{H1} we have

$$g_{H1} = \lambda^T \frac{\partial}{\partial \phi} (Rc) = \lambda^T \frac{\partial R}{\partial \phi} c = c^T \left(\frac{\partial R}{\partial \phi} \right)^T \lambda = c^T \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} R^T \lambda = (0 \ -c_3 \ c_2) R^T \lambda = g_{Hx}.$$

In a similar way we can show that

$$g_{H2} = g_{Hy} \cos\phi - g_{Hz} \sin\phi$$

$$g_{H3} = -g_{Hx} \sin\theta + \cos\theta (g_{Hy} \sin\phi + g_{Hz} \cos\phi)$$

and this proves the assertion. We note that the torques g_{H1}, g_{H2}, g_{H3} would be needed if the Lagrange's equations are used instead of Euler's equations.

Rather than eliminating the constraint force f_H using Equation (19) we may define it approximately and solve the six second-order Equations (16) and (17). For this we replace the Lagrange multiplier terms in Equation (22) by a potential function V:

$$L = T - V \qquad (23)$$

where V is given by

$$V = \frac{1}{2} \left[K_x (x_H - \alpha_1)^2 + K_y (y_H - \alpha_2)^2 + K_z (z_H - \alpha_3)^2 \right] \qquad (24)$$

and K_x, K_y, K_z are large positive constants (Ref. 7, pp. 516-518). The introduction of the potential function V means that the constraint Equation (15) is replaced by elastic springs with spring constants K_x, K_y and K_z. The components of the spring force is therefore given by

$$f_{Hx} = -\frac{\partial V}{\partial x_H} = -K_x (x_H - \alpha_1); \quad x_H = x_c + (Rc)_1$$

$$f_{Hy} = -\frac{\partial V}{\partial y_H} = -K_y (y_H - \alpha_2); \quad y_H = y_c + (Rc)_2$$

$$f_{Hz} = -\frac{\partial V}{\partial z_H} = -K_z (z_H - \alpha_3); \quad z_H = z_c + (Rc)_3.$$

Another way to approximate the constraint force is to define the dissipation function

$$f = \frac{1}{2} (K'_x \dot{x}_H^2 + K'_y \dot{y}_H^2 + K'_z \dot{z}_H^2) \qquad (25)$$

where K'_x, K'_y, K'_z are large positive constants and determine the components of the constraint force from the relations:

$$f_{Hx} = -\frac{\partial f}{\partial \dot{x}_H} = -K'_x \dot{x}_H$$

$$f_{Hy} = -\frac{\partial f}{\partial \dot{y}_H} = -K'_y \dot{y}_H$$

$$f_{Hz} = -\frac{\partial f}{\partial \dot{z}_H} = -K'_z \dot{z}_H.$$

In this way the constraint force is approximated by the viscous damping force. Of course, the constraint force may be determined by combining the elastic and damping force or by defining V and f in other ways than that done in Equations (24) and (25).

Rather than approximating the constraint force analytically through the use of penalty functions such as V and f we may approximate the constraint force numerically (Ref. 3) by solving the equations of motion and constraint equation simultaneously. To do this the equations are first written in the form

$$F\left(u, \dot{u}, t\right) = 0. \tag{26}$$

Now given $u_n \equiv u(t_n)$ it is required to find $u_{n+1} \equiv u(t_{n+1})$ where $t_n = nh$ and h is the time step. Using for example the backward Euler method Equation (26) becomes

$$F\left(u_{n+1}, \frac{u_{n+1} - u_n}{h}, t_{n+1}\right) = 0 \tag{27}$$

which is solved for u_{n+1} using Newton-Raphson method and sparse matrix techniques.

This method is not considered further and is mentioned here only for the sake of completeness.

3.2 Body with Universal Joint

Let us now consider the motion of a rigid body connected to a fixed body with a universal joint. The rigid body has now two rotational degrees of freedom. For definiteness sake let

$$\phi = 0. \tag{28}$$

Due to this constraint there will be a constraint torque g_H and Equation (17) must be replaced by

$$I\dot{\omega} + \omega \times I\omega = G + g_H + c \times R^T f_H. \tag{29}$$

In the body-fixed axes the unit axes of rotation are $p_1 = (-\sin\theta \quad 0 \quad \cos\theta)^T$ and $p_2 = (0 \quad 1 \quad 0)^T$. Since the constraint torque g_H, by definition, is perpendicular to both these axes of rotation we have

$$g_{Hz} \cos\theta - g_{Hx} \sin\theta = 0 \tag{30}$$

$$g_{Hy} = 0. \tag{31}$$

Equations (30) and (31) can also be obtained by the Lagrange multiplier method. For this let

$$L_1 = L + \lambda_4 \phi \tag{32}$$

where L is given by Equation (22). Due to the extra term $\lambda_4 \phi$ we have the constraint torque g_H such that

$$g_{H1} = \lambda_4, \quad g_{H2} = 0, \quad g_{H3} = 0. \tag{33}$$

Using Equation (14) with $\phi = 0$ we obtain

$$g_{Hx} = \lambda_4 \qquad g_{Hy} = 0 \qquad g_{Hz} = \lambda_4 \tan\theta. \tag{34}$$

Eliminating λ_4 from Equation (34) we have $g_{Hz} \cos\theta - g_{Hx} \sin\theta = 0$ and $G_{Hy} = 0$.

To obtain the equations of motion with the least number we must eliminate f_H and g_H from Equation (29). Eliminating f_H (see Eq. (21)) we have

$$I_H \dot{\omega} + \omega \times I_H \omega = G + g_H - c \times R^T F. \qquad (35)$$

To eliminate g_H we premultiply Equation (35) by the row matrices $p_1^T = (-\sin\theta \quad 0 \quad \cos\theta)$ and $p_2^T = (0 \quad 1 \quad 0)$:

$$p_1^T (I_H \dot{\omega} + \omega \times I_H \omega) = p_1^T (G - c \times R^T F) + p_1^T g_H$$

$$p_2^T (I_H \dot{\omega} + \omega \times I_H \omega) = p_2^T (G - c \times R^T F) + p_2^T g_H$$

From Equations (30) and (31) it follows that $p_1^T g_H = 0$ and $p_2^T g_H = 0$. Hence the two rotational equations are

$$p_1^T (I_H \dot{\omega} + \omega \times I_H \omega) = p_1^T (G - c \times R^T F) \qquad (36)$$

$$p^T (I_H \dot{\omega} + \omega \times I_H \omega) = p^T (G - c \times R^T F) \qquad (37)$$

where ω (Eq. (4) with $\phi = 0$) is given by

$$\omega = \begin{pmatrix} -\dot{\psi} \sin\theta \\ \dot{\theta} \\ \dot{\psi} \cos\theta \end{pmatrix}$$

Instead of eliminating the constraint torque g_H and constraint force f_H from Equation (29) (or g_H from Eq. (35)) we may determine them approximately and solve the six second-order Equations (16) and (29) (or three Eq. (35)). For this we can use the same procedures as discussed in Section 3.1. For example, the term $\lambda_4 \phi$ in Equation (32) may be replaced by the negative of a potential function V.

Let $V = \frac{1}{2} K_\phi \phi^2$ where K_ϕ is a large positive constant. Then

$$g_{H1} = -\frac{\partial V}{\partial \phi} = -K_\phi \phi, \quad g_{H2} = -\frac{\partial V}{\partial \theta} = 0, \quad g_{H3} = -\frac{\partial V}{\partial \psi} = 0. \qquad (38)$$

Using Equations (14) and (38) and assuming ϕ small we get

$$g_{Hx} = -K_\phi \phi, \quad g_{Hy} = 0, \quad g_{Hz} = -K_\phi \phi \tan\theta. \qquad (39)$$

We now show that this obvious choice of V is not good enough. From Equation (30) we see that $g_{Hx} = 0$ for $\theta = \pi/2$. However, g_{Hx} as defined in Equation (39) does not tend to zero as

$\theta \to \pi/2$ (unless $\phi = 0$). Moreover g_{Hz} becomes infinite as $\theta \to \pi/2$. To correct this situation we set

$$g_{Hx} = -(K_\phi \cos\theta)\phi, \quad g_{Hy} = 0, \quad g_{Hz} = -(K_\phi \sin\theta)\phi \tag{40}$$

which is obtained by defining $V = \dfrac{1}{2}(K_\phi \cos\theta)\phi^2$.

We remark that since f_H is defined in terms of r_H or its derivative it may be computationally advantageous to replace Equation (16) by its equivalent form

$$m\,\frac{d^2 r_H}{dt^2} = F + mR(\dot\omega \times c + \omega \times (\omega \times c)) + f_H \tag{41}$$

which is obtained by substituting the value of $\dfrac{d^2 r_C}{dt^2}$ from Equation (18) into Equation (16).

4.0 MULTI-BODY EQUATIONS OF MOTION

In this section we derive the equations of motion of a system of n rigid bodies connected together by hinges in an open chain such that each body is connected directly to at most two rigid bodies.

4.1 Systems with Ball-and-Socket Joints

First we consider the case when all hinges are ball-and-socket joints. The relative motion of any two neighbouring bodies is therefore a pure rotation with three degrees of freedom. Consider three neighbouring bodies i−1, i and i+1 in the chain as shown in Figure 3. Let $C_i H_{j-1} = L_{i,i-1}$, $C_i H_i = L_{i,i+1}$.
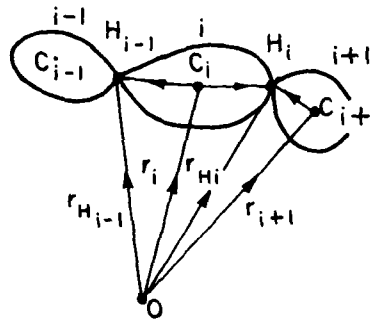

FIG. 3

As in Equations (16) and (17) the equations of motion of body i can be written as:

$$m_i \frac{d^2 r_i}{dt^2} = F^i + f_H^{i,i-1} + f_H^{i,i+1} \tag{42}$$

$$I_i \dot\omega_i + \omega_i \times I_i \omega_i = G^i + L_{i,i-1} \times (R^i)^T f_H^{i,i-1} + L_{i,i+1} \times (R^i)^T f_H^{i,i+1} \tag{43}$$

where $f_H^{i,i-1}$ and $f_H^{i,i+1}$ are the constraint forces on body i at the hinges H_{i-1} and H_i; F^i is the sum of external forces acting on body i; G^i is the sum of external torques; m_i, I_i and ω_i respectively are the mass, inertia matrix and the absolute angular velocity of body i; $r_i = OC_i$ and R^i is the transformation matrix from body i to the inertial frame. By summing the translational equations from i = 1 to n we obtain the equation of motion of the centre of mass of the system:

$$m \frac{d^2 r}{dt^2} = \sum_{i=1}^{n} F^i \tag{44}$$

where

$$m = \sum_{i=1}^{n} m_i \quad \text{is the total system mass,} \tag{45}$$

$$r = \frac{1}{m} \sum_{i=1}^{n} m_i r_i \quad \text{is the system centre of mass.} \tag{46}$$

In deriving Equation (44) we have used the fact that the constraint force on body i at hinge H_i is equal and opposite to the constraint force on body i + 1 i.e.

$$f^{i+1,i} = -f^{i,i+1} \qquad i = 1,2,\ldots,n-1. \tag{47}$$

Also $f_H^{1,0} = 0 = f_H^{n,n+1}$.

The motion of the system can now be determined from Equations (43) and (44) provided the constraint forces are known.

4.1.1 Determination of Constraint Forces

As in the case of a single rigid body we shall obtain first the equations of motion by eliminating the constraint forces. The constraint equations at the hinges H_i, i = 1,2,..., n–1 are

$$r_{i+1} + R^{i+1} L_{i+1,i} = r_i + R^i L_{i,i+1}; \quad i = 1,2,\ldots,n-1. \tag{48}$$

From the translational equations of bodies i and i + 1 we have

$$\frac{d^2 r_i}{dt^2} = \frac{1}{m_i} (F^i + f_H^{i,i-1} + f_H^{i,i+1}) \tag{49}$$

$$\frac{d^2 r_{i+1}}{dt^2} = \frac{1}{m_{i+1}} (F^{i+1} + f_H^{i+1,i} + f_H^{i+1,i+2}). \tag{50}$$

Subtracting Equation (50) from Equation (49) and using Equation (47) we obtain

$$-\frac{1}{m_i} f_H^{i-1,i} + \left(\frac{1}{m_i} + \frac{1}{m_{i+1}}\right) f^{i,i+1} - \frac{1}{m_{i+1}} f_H^{i+1,i+2} = d_i; \quad i = 1,2,\ldots,n-1 \tag{51}$$

where

$$d_i = \frac{d^2 r_i}{dt^2} - \frac{d^2 r_{i+1}}{dt^2} + \left(\frac{1}{m_{i+1}} F^{i+1} - \frac{1}{m_i} F^i\right). \tag{52}$$

Equation (51) with $f_H^{0,1} = 0 = f_H^{n,n+1}$ represents a tridiagonal system and can be inverted easily by using Thomas algorithm. Let A denote the inverse of the matrix representing the tridiagonal system of equations. Then

$$f_H^{i,i+1} = \sum_{j=1}^{n-1} a_{ij} d_j; \quad i = 1, 2, \ldots, n-1. \tag{53}$$

Our next task is to express d_i in terms of $L_{i,i+1}$ and $L_{i+1,i}$. Differentiating the constraint Equation (48) twice with respect to time we obtain

$$\frac{d^2 r_i}{dt^2} - \frac{d^2 r_{i+1}}{dt^2} = R^{i+1} \left(\dot{\omega}_{i+1} \times L_{i+1,i} + \omega_{i+1} \times (\omega_{i+1} \times L_{i+1,i})\right)$$

$$- R^i \left(\dot{\omega}_i \times L_{i,i+1} + \omega_i \times (\omega_i \times L_{i,i+1})\right). \tag{54}$$

Substituting Equation (54) into Equation (52) we obtain

$$d_i = R^{i+1} \left(\dot{\omega}_{i+1} \times L_{i+1,i} + \omega_{i+1} \times (\omega_{i+1} \times L_{i+1,i})\right) - R^i\left(\dot{\omega}_i \times L_{i,i+1}\right.$$

$$\left. + \omega_i \times (\omega_i \times L_{i,i+1})\right) + \left(\frac{1}{m_{i+1}} F^{i+1} - \frac{1}{m_i} F^i\right); \quad i = 1, 2, \ldots, n-1. \tag{55}$$

Thus the constraint forces can be determined from Equations (53) and (55). We note that Equation (19) is a special case of Equation (53) with $n = 2$, $m_1 = \infty$, $\omega_1 = 0$, $f_H = -f_H^{1,2}$. We also mention that this method of deriving the constraint forces is different from that used in References 1 and 2.

4.1.2 Elimination of Constraint Forces

To eliminate the constraint forces from the rotational Equation (43) we need to evaluate the expression

$$L_{i,i-1} \times (R^i)^T f_H^{i,i-1} + L_{i,i+1} \times (R^i)^T f_H^{i,i+1} \tag{56}$$

Substituting Equation (53) into Equation (56) we obtain

$$-L_{i,i-1} \times (R^i)^T \sum_{j=1}^{n-1} a_{i-1,j} d_j + L_{i,i+1} \times (R^i)^T \sum_{j=1}^{n-1} a_{ij} d_j. \tag{57}$$

From Equation (55) we see that the terms containing $\dot{\omega}_j$ are:

$$L_{i,i-1} \times R^{ij} \left[a_{i-1,j} \, \dot{\omega}_j \times L_{j,j+1} - a_{i-1,j-1} \dot{\omega}_j \times L_{j,j-1} \right]$$

$$- L_{i,i+1} \times R^{ij} \left[a_{ij} \, \dot{\omega}_j \times L_{j,j+1} - a_{i,j-1} \dot{\omega}_j \times L_{j,j-1} \right]$$

(58)

where $R^{ij} = (R^i)^T R^j$ denotes the transformation matrix from reference frame j to i. Using the identity $A \times (B \times C) = (A \cdot C)B - (A \cdot B)C$ the first term in Equation (58) can be rewritten as

$$L_{i,i-1} \times R^{ij}(a_{i-1,j}\dot{\omega}_j \times L_{j,j+1}) = a_{i-1,j} \left[(L_{i,i-1} \cdot R^{ij}L_{j,j+1})R^{ij}\dot{\omega}_j - (L_{i,i-1} \cdot R^{ij}\dot{\omega}_j)R^{ij}L_{j,j+1} \right]$$

$$= a_{i-1,j}(L_{i,i-1}^T \, R^{ij}L_{j,j+1} R^{ij}\dot{\omega}_j - R^{ij}L_{j,j+1}L_{i,i-1}^T R^{ij}\dot{\omega}_j)$$

$$= a_{i-1,j}R^{ij}(R^{ji}L_{i,i-1}^T R^{ij}L_{j,j+1} E - L_{j,j+1}L_{i,i-1}^T)R^{ij} \dot{\omega}_j$$

(59)

Hence Equation (58) can be simplified as

$$-R^{ij} \, \check{K}_{ij} R^{ij}\dot{\omega}_j$$

(60)

where the matrix $\check{K}_{ij}$ (i,j=1, . . . ,n) is given by

$$\check{K}_{ij} = -R^{ji} \left[L_{i,i-1}^T \, R^{ij} \, (a_{i-1,j}L_{j,j+1} - a_{i-1,j-1} L_{j,\,j-1}) \right.$$

$$\left. - L_{i,i+1}^T R^{ij}(a_{ij}L_{j,j+1} - a_{i,j-1} L_{j,j-1}) \right] E$$

$$+ (a_{i-1,j}L_{j,j+1} - a_{i-1,j-1} L_{j,j-1})L_{i,i-1}^T$$

$$- (a_{ij}L_{j,j+1} - a_{i,j-1} L_{j,j-1})L_{i,i+1}^T$$

(61)

with $L_{1,0} = 0 = L_{n,n+1}$. Using the fact that $a_{ij} = a_{ji}$ it is easy to see that $\check{K}_{ij}^T = \check{K}_{ji}$.

Similarly combining the ω_j and F^j terms in Equation (57) we get

$$-R^{ij}\omega_j \times \check{K}_{ji}\omega_j + N_{ij} + u_{ij}$$

(62)

where

$$N_{ij} = -\omega_j^T \omega_j \left[(a_{i,j-1} L_{i,i+1} - a_{i-1,j-1} L_{i,i-1}) \times R^{ij}L_{j,j-1} \right.$$

$$\left. - (a_{ij}L_{i,i+1} - a_{i-1,j}L_{i,i-1}) \times R^{ij}L_{j,j+1} \right]$$

(63)

$$u_{ij} = \frac{1}{m_j} \left[(a_{i-1,j} - a_{i-1,j-1})L_{i,i-1} - (a_{ij} - a_{i,j-1}) \right.$$

$$\left. L_{i,i+1} \right] \times (R^i)^T \, F^j.$$

(64)

Using Equations (60) and (62), the rotational Equation (43) for body i, i = 1,2, . . . ,n, can be written as

$$\sum_{j=1}^{n} R^{ij} K_{ij} R^{ij} \dot{\omega}_j + \sum_{j=1}^{n} R^{ij} \omega_j \times K_{ji} \omega_j = G^i + \sum_{j=1}^{n} (N_{ij} + u_{ij}) \tag{65}$$

where

$$K_{ij} = \hat{K}_{ij}, i \neq j$$

$$= I_i + \hat{K}_{ii}, i = j.$$

Equations (44) and (65) give the required 3n+3 scalar equations of motion.

4.1.3 Approximate Determination of Constraint Forces

Instead of determining the constraint forces $f_H^{i,i+1}$ from Equation (53) exactly we shall now use approximate methods discussed in Section 3.1 to determine these forces. Let

$$r_{Hi}^- = r_i + R^i L_{i, i+1}$$

$$r_{Hi}^+ = r_{i+1} + R^{i+1} L_{i+1, i}.$$

Then the constraint equation at the hinge H_i (see Eq. (48)) can be written as:

$$r_{Hi}^+ - r_{Hi}^- = 0.$$

Differentiating r_{Hi}^+ twice with respect to t we get

$$\frac{d^2 r_{Hi}^+}{dt^2} = \frac{d^2 r_{i+1}}{dt^2} + R^{i+1} \left(\dot{\omega}_{i+1} \times L_{i+1, i} + \omega_{i+1} \times (\omega_{i+1} \times L_{i+1, i}) \right).$$

Using Equation (42) we obtain the following differential equations for r_{Hi}^+, i = 1,2. . . ,n−1:

$$m_{i+1} \frac{d^2 r_{Hi}^+}{dt^2} = F^{i+1} + m_{i+1} R^{i+1} \left(\dot{\omega}_{i+1} \times L_{i+1,i} + \omega_{i+1} \times (\omega_{i+1} \times L_{i+1,i}) \right) + f_H^{i+1,i} + f_H^{i+1,i+2}. \tag{66}$$

To determine the constraint forces approximately we define the potential function

$$V_i = \frac{1}{2} K_i \left(r_{Hi}^+ - r_{Hi}^- \right)^T \left(r_{Hi}^+ - r_{Hi}^- \right) \tag{67}$$

where K_i is a large positive constant. The constraint force $f_H^{i+1,i}$ is therefore given by

$$f_H^{i+1,i} \approx - \frac{\partial V_i}{\partial r_{Hi}^+} = -K_i \left(r_{Hi}^+ - r_{Hi}^- \right), \quad i = 1,2, \ldots ,n-1 \tag{68}$$

where

$$r_{Hi}^- = r_{Hi-1}^+ + R^i (L_{i,i+1} - L_{i,i-1}), i = 2,3,\ldots,n-1$$

and r_{H1}^- is obtained by solving the differential equation

$$m_1 \frac{d^2 r_{H1}^-}{dt^2} = F^1 + m_1 R^1 \left(\dot{\omega}_1 \times L_{1,2} + \omega_1 \times (\omega_1 \times L_{1,2}) \right) + f_H^{1,2}. \tag{69}$$

Equations (43), (66) and (69) with constraint forces $f_H^{i+1,i}$, $i = 1,2,\ldots,n-1$ given by Equation (68) determine the motion of the multi-body system. Comparing these equations with those in Equation (65) we see that although the number of differential equations to be solved is more, they are less densely coupled and may be as convenient to solve on a computer as the more densely coupled Equation (65).

We mention that the constraint forces may also be determined by defining the dissipation function

$$f_i = \frac{1}{2} K_i' \left(\dot{r}_{Hi}^+ - \dot{r}_{Hi}^- \right)^T \left(\dot{r}_{Hi}^+ - \dot{r}_{Hi}^- \right) \tag{70}$$

where K_i' is a large positive constant. The force $f_H^{i+1,i}$ is now given by

$$f_H^{i+1,i} = - \frac{\partial f_i}{\partial \dot{r}_{Hi}^+} = -K_i' \left(\dot{r}_{Hi}^+ - \dot{r}_{Hi}^- \right).$$

4.2 Systems with Ball-and-Socket Joints, Universal Joints and Pin Joints

We now consider the case when the rotational degrees of freedom at some joints may be less than three i.e. the hinges may be universal (two degrees of freedom) or pin (one degree of freedom) joints. Due to universal and pin joints at the hinges the equations of motion (42) and (43) must be replaced by

$$m_i \frac{d^2 r_i}{dt^2} = F^i + f_H^{i,i-1} + f_H^{i,i+1} \tag{71}$$

$$I_i \dot{\omega}_i + \omega_i \times I_i \omega_i = G^i + g_H^{i,i-1} - R^{i,i+1} g_H^{i+1,i} + L_{i,i-1} \times (R^i)^T f_H^{i,i-1}$$

$$+ L_{i,i+1} \times (R^i)^T f_H^{i,i+1}. \tag{72}$$

where $g_H^{i,i-1}$ denotes the constraint torque on body i at the hinge i-1 expressed in the ith body axes. Of course, $g_H^{1,0} = 0 = g_H^{n+1,n}$.

To determine the orientation of the bodies we need to eliminate not only the unknown constraint forces but also the unknown torques. The unknown constraint forces can be eliminated in the same way as was discussed in Section 4.1. The rotational equations can therefore be written as (see Eq. (65)):

$$\sum_{j=1}^{n} R^{ij} K_{ij} R^{ij} \dot{\omega}_j + \sum_{j=1}^{n} R^{ij} \omega_j \times K_{ji} \omega_j = G^i + g_H^{i,i-1} - R^{i,i+1} g_H^{i+1,i} + \sum_{j=1}^{n} (N_{ij} + u_{ij}). \tag{73}$$

$i = 1, 2, \ldots, n$. Multiplying Equation (73) by R^{1i} and summing the equations from $i = 1$ to n we obtain

$$\sum_{i=1}^{n} R^{1i} \sum_{j=1}^{n} R^{ij} K_{ij} R^{ij} \dot{\omega}_j + \sum_{i=1}^{n} R^{1i} \sum_{j=1}^{n} R^{ij} \omega_j \times K_{ji} \omega_j = \sum_{i=1}^{n} R^{1i} \left(G^i + \sum_{j=1}^{n} (N_{ij} + u_{ij}) \right) \tag{74}$$

since $\sum\limits_{i=1}^{n} R^{1i} \left(g_H^{i,i-1} - R^{i,i+1} g_H^{i+1,i} \right) = g_H^{1,0} = 0.$

4.2.1 Determination of Constraint Torques

We now determine the constraint torques $g_H^{i,i-1}$, $i = 2, 3, \ldots, n$. For this we add all the rotational equations except the first $i-1$ equations. From Equation (73) we get

$$\sum_{k=i}^{n} R^{ik} \sum_{j=1}^{n} R^{kj} K_{kj} R^{kj} \dot{\omega}_j + \sum_{k=i}^{n} \sum_{j=1}^{n} R^{kj} \omega_j \times K_{jk} \omega_j = \sum_{k=i}^{n} R^{ik} G^k + \sum_{k=i}^{n} R^{ik} \sum_{j=1}^{n} (N_{kj} + u_{kj})$$

$$+ \sum_{k=i}^{n} R^{ik} \left(g_H^{k,k-1} - R^{k,k+1} g_H^{k+1,k} \right). \tag{75}$$

Since $\sum\limits_{k=i}^{n} R^{ik} \left(g_H^{k,k-1} - R^{k,k+1} g_H^{k+1,k} \right) = g_H^{i,i-1}$ we have

$$g_H^{i,i-1} = \sum_{k=i}^{n} R^{ik} \sum_{j=1}^{n} R^{kj} K_{kj} R^{kj} \dot{\omega}_j + \sum_{k=i}^{n} \sum_{j=1}^{n} R^{kj} \omega_j \times K_{jk} \omega_j - \sum_{k=i}^{n} R^{ik} \left(G^k + \sum_{j=1}^{n} (N_{kj} + u_{kj}) \right),$$

$$i = 2, 3, \ldots, n. \tag{76}$$

4.2.2 Elimination of Constraint Torques

In the presence of universal and pin joints the relative angular velocity of two neighbouring bodies can be defined in terms of fewer angular rates — two in the case of a universal joint and one for a pin joint. Let Ω_i denote the relative angular velocity of body i relative to body $i-1$.

Then

$$\omega_i = R^{i,i-1} \omega_{i-1} + \Omega_i \tag{77}$$

where

$$\Omega_i = p_{i1} \dot{\gamma}_{i1} + p_{i2} \dot{\gamma}_{i2} + p_{i3} \dot{\gamma}_{i3} = \sum_{j=1}^{n_i} p_{ij} \dot{\gamma}_{ij}, i = 2, 3, \ldots, n, \tag{78}$$

$n_i = 1, 2$ or 3 according as the hinge is a pin, universal or ball-and-socket joint and p_{ij} are unit vectors along the axes of rotation. For example in the case of universal joint we may take $\dot{\gamma}_{i1} = \dot{\psi}_i, \dot{\gamma}_{i2} = \dot{\theta}_i$.

From Equation (4) with $\dot{\phi} = 0$ it follows that $p_{i1} = (-\sin\theta_i \quad 0 \quad \cos\theta_i)^T$ and $p_{i2} = (0 \quad 1 \quad 0)^T$. Using Equation (77) we can express ω_i in terms of ω_1 and the relative angular rates $\dot{\gamma}_{ij}$:

$$\omega_i = R^{i,i-1} \omega_{i-1} + \Omega_i$$

$$= R^{i,i-1}(R^{i-1,i-2}\omega_{i-2} + \Omega_{i-1}) + \Omega_i$$

$$= R^{i,i-2} \omega_{i-2} + R^{i,i-1} \Omega_{i-1} + \Omega_i$$

since $R^{i,i-1} R^{i-1,i-2} = R^{i,i-2}$. Continuing the above process we obtain

$$\omega_i = R^{i1} \omega_1 + \sum_{j=2}^{n} R^{ij} \Omega_j. \tag{79}$$

In the same way $\dot{\omega}_i$ can be expressed in terms of $\dot{\omega}_1$ and $\ddot{\gamma}_{ij}$. Differentiating Equation (77) we get

$$\dot{\omega}_i = R^{i,i-1} \dot{\omega}_{i-1} + p_{i1}\ddot{\gamma}_{i1} + p_{i2}\ddot{\gamma}_{i2} + p_{i3}\ddot{\gamma}_{i3} + v_i \tag{80}$$

where v_i is given by

$$v_i = -\Omega_i \times R^{i,i-1} \omega_{i-1} + \dot{p}_{i1}\dot{\gamma}_{i1} + \dot{p}_{i2}\dot{\gamma}_{i2} + \dot{p}_{i3}\dot{\gamma}_{i3}.$$

Using Equation (80) recursively we obtain

$$\dot{\omega}_i = R^{i1} \dot{\omega}_1 + \sum_{j=2}^{i} R^{ij}(p_{j1}\ddot{\gamma}_{j1} + p_{j2}\ddot{\gamma}_{j2} + p_{j3}\ddot{\gamma}_{j3}) + \sum_{j=2}^{i} R^{ij}v_j. \tag{81}$$

The rotational degrees of freedom $n' = \sum_{i=2}^{n} n_i + 3$. Therefore, to determine the orientation of the multi-body system we require n' equations. Equation (74) with ω_i and $\dot{\omega}_i$ defined by Equations (79) and (81) provides three equations. To determine the rest of the equations we use Equation (76). By definition, the constraint torque is perpendicular to the axes of rotation and hence we have:

$$p_{ih_i}^T g_H^{i,i-1} = 0, \quad i = 2,3,\ldots,n, \quad h_i = 1,\ldots,n_i. \tag{82}$$

Substituting the value of $g_H^{i,i-1}$ from Equation (76) we obtain the remaining $n'-3$ equations:

$$p_{ih_i}^T \left[\sum_{k=i}^{n} R^{ik} \sum_{j=1}^{n} R^{kj}K_{kj}R^{kj}\dot{\omega}_j + \sum_{k=i}^{n} \sum_{j=1}^{n} R^{kj}\omega_j \times K_{jk}\omega_j \right.$$

$$\left. - \sum_{k=i}^{n} R^{ik} \left(G^k + \sum_{j=1}^{n} (N_{kj} + u_{kj}) \right) \right] = 0. \tag{83}$$

Substituting the value of $\dot{\omega}_j$ from Equation (81) into Equations (74) and (83), the equations of motion can be rewritten in the following form:

$$b_{11}\dot{\omega}_1 + \sum_{k=2}^{n} \left(b_{1k}^{1} \ddot{\gamma}_{k1} + \ldots + b_{1k}^{n_k} \ddot{\gamma}_{kn_k} \right) = f_1, \tag{84}$$

$$b_{i1}^{h_i}\dot{\omega}_1 + \sum_{k=2}^{n} \left(b_{ikh_i}^{1} \ddot{\gamma}_{k1} + \ldots + b_{ikh_i}^{n_k} \ddot{\gamma}_{kn_k} \right) = f_i^{h_i}, i = 2,3,\ldots,n \tag{85}$$

$$h_i = 1, \ldots, n_i,$$

where

$$b_{11} = \sum_{i=1}^{n} \sum_{j=1}^{n} R^{1j} K_{ij} R^{i1} \qquad (3 \times 3 \text{ matrix})$$

$$b_{1k}^{h_k} = \sum_{j=k}^{n} \sum_{i=1}^{n} R^{1j} K_{ij} R^{ik} p_{kh_k} \qquad (3 \times 1 \text{ matrix})$$

$$f_1 = \sum_{i=1}^{n} R^{1i} \left(G^i + \sum_{j=1}^{n} (N_{ij} + u_{ij}) \right) - \sum_{i=1}^{n} R^{1i} \sum_{j=1}^{n} R^{ij} \omega_j \times K_{ji} \omega_j$$

$$- \sum_{k=2}^{n} \left(\sum_{j=k}^{n} \sum_{i=1}^{n} R^{1j} K_{ij} R^{ik} \right) v_k \qquad (3 \times 1 \text{ matrix})$$

$$b_{i1}^{h_i} = \left(b_{1i}^{h_i} \right)^T \qquad (1 \times 3 \text{ matrix})$$

$$b_{ikh_i}^{h_k} = p_{ih_i}^{T} \sum_{j=k}^{n} \sum_{r=i}^{n} R^{ij} K_{rj} R^{rk} p_{kh_k} \qquad (\text{scalar})$$

$$f_i^{h_i} = p_{ih_i}^{T} \left[\sum_{k=i}^{n} R^{ik} \left(G^k + \sum_{j=1}^{n} (N_{kj} + u_{kj}) \right) - \sum_{k=i}^{n} \sum_{j=1}^{n} R^{kj} \omega_j \times K_{jk} \omega_j \right.$$

$$\left. - \sum_{r=i}^{n} \sum_{j=1}^{n} R^{ij} K_{rj} \sum_{k=2}^{j} R^{rk} v_k \right] \qquad (\text{scalar})$$

and ω_j can be evaluated by using the recursive Equation (77). For computing the cross products appearing in the above equations the relation $a \times b = \tilde{a} \, b$ can be used.

4.2.3 Approximate Determination of Constraint Torques

In this section we shall discuss procedures for determining approximately the constraint torques. Assume first that body i is connected to body i-1 by a pin joint. Let p_i denote the unit vector along the axes of rotation of body i relative to body i-1.

Then

$$R^{i,i-1} p_i = p_i \tag{86}$$

where $R^{i,i-1}$ is the transformation matrix connecting the components of a vector expressed in frame i-1 to frame i and is given by

$$R^{i,i-1} = (R^i)^{-1} R^{i-1} \tag{87}$$

Substituting Equation (87) into Equation (86) and multiplying both sides by R^i we obtain

$$R^{i-1} p_i = R^i p_i \tag{88}$$

The constraint Equation (88) gives three scalar equations (only two are independent). Let the three scalar equations be written as

$$E_{i1} = 0, \quad E_{i2} = 0, \quad E_{i3} = 0. \tag{89}$$

As in the case of a single body we now define a potential function V_i by the relation

$$V_i = \frac{1}{2} K_i \left(E_{i1}^2 + E_{i2}^2 + E_{i3}^2 \right) \tag{90}$$

and determine $g_{H1}^{i,i-1}, g_{H2}^{i,i-1}, g_{H3}^{i,i-1}$ by the relations

$$g_{H1}^{i,i-1} = -\frac{\partial V_i}{\partial \phi_i}, \quad g_{H2}^{i,i-1} = -\frac{\partial V_i}{\partial \theta_i}, \quad g_{H3}^{i,i-1} = -\frac{\partial V_i}{\partial \psi_i} \tag{91}$$

Having determined these, the components of the constraint torque in the body frame i can be determined using Equation (14):

$$g_{Hx}^{i,i-1} = g_{H1}^{i,i-1}$$

$$g_{Hy}^{i,i-1} = g_{H2}^{i,i-1} \cos\phi_i + \frac{\sin\phi_i}{\cos\theta_i} \left(g_{H1}^{i,i-1} \sin\theta_i + g_{H3}^{i,i-1} \right) \tag{92}$$

$$G_{Hz}^{i,i-1} = -g_{H2}^{i,i-1} \sin\phi_i + \frac{\cos\phi_i}{\cos\theta_i} \left(g_{H1}^{i,i-1} \sin\theta_i + g_{H3}^{i,i-1} \right).$$

It should be noted that these components are not required if the rotational equations are written in the Lagrange form (see Eq. (11)).

We can also express the constraint condition in terms of the Euler angle rates. For this let Ω_i denote the relative angular velocity of body i with respect to body i-1. Then there exist two unit vectors q_{i1}, q_{i2} orthogonal to each other such that

$$q_{i1}^T \Omega_i = 0, \quad q_{i2}^T \Omega_i = 0 \tag{93}$$

i.e. the relative angular velocity Ω_i is orthogonal to the constraint axes q_{i1} and q_{i2}.

To determine the constraint torque by using Equation (93) we form the dissipation function f_i:

$$f_i = \frac{1}{2} K_i' \left(\left(q_{i1}^T \Omega_i \right)^2 + \left(q_{i2}^T \Omega_i \right)^2 \right) \tag{94}$$

and obtain

$$g_{H1}^{i,i-1} = - \frac{\partial f_i}{\partial \dot{\phi}_i}, \quad g_{H2}^{i,i-1} = - \frac{\partial f_i}{\partial \dot{\theta}_i}, \quad g_{H3}^{i,i-1} = - \frac{\partial f_i}{\partial \dot{\psi}_i} \tag{95}$$

The components of the torque in frame i can now be obtained from Equations (92) and (95).

Let us now consider the case of a universal joint. In this case there is a vector q_i such that

$$q_i^T \, \Omega_i = 0 \tag{96}$$

The constraint torque can be determined as above by defining the dissipation function f_i:

$$f_i = \frac{1}{2} K_i' \left(q_i^T \, \Omega_i \right)^2 \tag{97}$$

Example

As an example to the application of determining constraint torques approximately we consider the case of a remote manipulator system with six degrees of freedom — a universal joint at the shoulder, a pin joint at the elbow and a ball-and-socket joint at the wrist (Ref. 8).
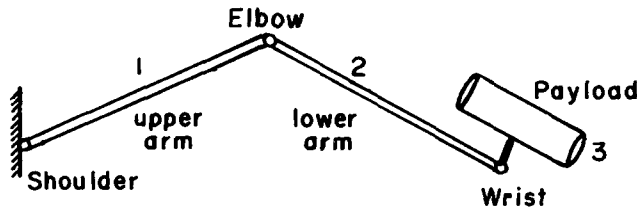

FIG. 4

Assuming the links are rigid it can be treated as a three-body system — upper arm, lower arm and the payload. (In the terminology used here it should be regarded as a four-body system with the first body held stationary.) Let the axes of rotations at the shoulder be z and y axes and that at the elbow be y axis. Expressing $\Omega_1 = \omega_1$ and q_1 in inertial frame and using constraint Equation (96) we have

$$\left(\cos\psi_1 \quad \sin\psi_1 \quad 0 \right) \begin{pmatrix} \dot{\phi}_1 \cos\psi_1 \cos\theta_1 - \dot{\theta}_1 \sin\psi_1 \\ \dot{\phi}_1 \sin\psi_1 \cos\theta_1 + \dot{\theta}_1 \cos\psi_1 \\ \dot{\psi}_1 - \dot{\phi}_1 \sin\theta_1 \end{pmatrix} = 0$$

which yields

$$\dot{\phi}_1 \cos\theta_1 = 0 \tag{98}$$

To find the constraint conditions at the elbow we apply Equation (88) with $p_2^T = (0 \quad 1 \quad 0)$ and obtain

$$E_{21} \equiv \cos\psi_2 \sin\theta_2 \sin\phi_2 \ - \ \sin\psi_2 \cos\phi_2 \ - \ \cos\psi_1 \sin\theta_1 \sin\phi_1 \ + \ \sin\psi_1 \cos\phi_1 \ = 0$$

$$E_{22} \equiv \sin\psi_2 \sin\theta_2 \sin\phi_2 \ + \ \cos\psi_2 \cos\phi_2 \ - \ \sin\psi_1 \sin\theta_1 \sin\phi_1 \ - \ \cos\psi_1 \cos\phi_1 \ = 0$$

$$E_{23} \equiv \cos\theta_2 \sin\phi_2 \ - \ \cos\theta_1 \sin\phi_1 \ = 0 \tag{99}$$

Let us first determine the constraint torque at the shoulder. For this we form the dissipation function

$$f_1 \ = \frac{1}{2} \ K' \left(\dot\phi_1 \cos\theta_1 \right)^2$$

and obtain

$$g_{H1}^S \ = - \ \frac{\partial f_1}{\partial \dot\phi_1} \ = - \ K' \dot\phi_1 \cos^2\theta_1 \,, \quad g_{H2}^S \ = - \ \frac{\partial f_1}{\partial \dot\theta_1} \ = 0, \quad g_{H3}^S \ = - \ \frac{\partial f_1}{\partial \dot\psi_1} \ = 0,$$

where the superscript S stands for the shoulder. Using Equation (92) we get

$$g_{Hx}^S \ = - \ K' \dot\phi_1 \cos^2\theta_1 \,, \quad g_{Hy}^S \ = - \ K' \dot\phi_1 \sin\theta_1 \cos\theta_1 \sin\phi_1 \,, \quad g_{Hz}^S \ = - \ K' \dot\phi_1 \sin\theta_1 \cos\theta_1 \cos\phi_1$$

From Equation (98) it follows that $\dot\phi_1 = 0$ and hence $\phi_1 = $ constant $= 0$. Therefore for small ϕ_1, $\dot\phi_1$ the torque can be obtained from the relations:

$$g_{Hx}^S \ = - \ K' \dot\phi_1 \cos^2\theta_1 \,, \quad g_{Hy}^S \ = \ 0, \quad g_{Hz}^S \ = - \ K' \dot\phi_1 \sin\theta_1 \cos\theta_1$$

or (dividing by $\cos\theta_1$) from the equations:

$$g_{Hx}^S \ = - \ K' \dot\phi_1 \cos\theta_1 \,, \quad g_{Hy}^S \ = \ 0, \quad g_{Hz}^S \ = - \ K' \dot\phi_1 \sin\theta_1 \,. \tag{100}$$

We can also determine the torque by forming the potential function $V_1 = \frac{1}{2} \ (K\cos\theta_1)\phi_1^2$. In this way we have

$$g_{Hx}^S \ = - \ K\phi_1 \cos\theta_1 \,, \quad g_{Hy}^S \ = \ 0, \quad g_{Hz}^S \ = - \ K\phi_1 \sin\theta_1 \,. \tag{101}$$

Of course, the constraint torque can be determined by combining Equations (100) and (101).

We now determine the constraint torque at the elbow. For this we define

$$V_2 \ = \frac{1}{2} \ K \left(E_{21}^2 \ + \ E_{22}^2 \ + \ E_{23}^2 \right) \tag{102}$$

Differentiating partially with regard to ϕ_2, θ_2 and ψ_2 we obtain

$$g_{H1}^{2E} = -\frac{\partial V_2}{\partial \phi_2} = -K \left[E_{21} \frac{\partial E_{21}}{\partial \phi_2} + E_{22} \frac{\partial E_{22}}{\partial \phi_2} + E_{23} \frac{\partial E_{23}}{\partial \phi_2} \right]$$

$$g_{H2}^{2E} = -\frac{\partial V_2}{\partial \theta_2} = -K \left[E_{21} \frac{\partial E_{21}}{\partial \theta_2} + E_{22} \frac{\partial E_{22}}{\partial \theta_2} + E_{23} \frac{\partial E_{23}}{\partial \theta_2} \right] \qquad (103)$$

$$g_{H3}^{2E} = -\frac{\partial V_2}{\partial \psi_2} = -K \left[E_{21} \frac{\partial E_{21}}{\partial \psi_2} + E_{22} \frac{\partial E_{22}}{\partial \psi_2} + E_{23} \frac{\partial E_{23}}{\partial \psi_2} \right]$$

where g_{Hi}^{2E}, $i = 1,2,3$ denote the components of the torque on body 2 at the elbow joint. The torque components on body 1 at the elbow can be similarly obtained by differentiating V_2 partially with respect to ϕ_1, θ_1, ψ_1.

Since $\phi_1 = 0$ it follows from Equation (99) that $\phi_2 = 0$ and $\psi_2 - \psi_1 = 0$. Substituting the values of E_{21}, E_{22}, E_{23} from Equation (99) into the first equation of Equation (103) we obtain

$$g_{H1}^{2E} = -K \left[(\cos\psi_2 \sin\theta_2 \sin\phi_2 - \sin\psi_2 \cos\phi_2 - \cos\psi_1 \sin\theta_1 \sin\phi_1 + \sin\psi_1 \cos\phi_1) \right.$$

$$(\cos\psi_2 \sin\theta_2 \cos\phi_2 + \sin\psi_2 \sin\phi_2) + (\sin\psi_2 \sin\theta_2 \sin\phi_2 + \cos\psi_2 \cos\phi_2$$

$$- \sin\psi_1 \sin\theta_1 \sin\phi_1 - \cos\psi_1 \cos\phi_1) (\sin\psi_2 \sin\theta_2 \cos\phi_2 - \cos\psi_2 \sin\phi_2)$$

$$\left. + (\cos\theta_2 \sin\phi_2 - \cos\theta_1 \sin\phi_1) \cos\theta_2 \cos\phi_2 \right].$$

Using small angle approximations it can be seen that the above expression can be written as

$$g_{H1}^{2E} = K \left[(\psi_2 - \psi_1)\sin\theta_2 + \phi_1 \cos(\theta_2 - \theta_1) - \phi_2 \right]$$

Similarly, we have

$$g_{H2}^{2E} = 0$$

$$g_{H3}^{2E} = -K \left[(\psi_2 - \psi_1) + \phi_1 \sin\theta_1 - \phi_2 \sin\theta_2 \right].$$

Substituting these values into Equation (92) we obtain the components of the torque in frame 2:

$$g_{Hx}^{2E} = K \left[(\psi_2 - \psi_1)\sin\theta_2 + \phi_1 \cos(\theta_2 - \theta_1) - \phi_2 \right]$$

$$g_{Hy}^{2E} = 0 \qquad (104)$$

$$g_{Hz}^{2E} = -K \left[(\psi_2 - \psi_1)\cos\theta_2 - \phi_1 \sin(\theta_2 - \theta_1) \right].$$

The components of the constraint torque on body 1 expressed in body frame 1 can be similarly found. These are given by

$$g_{Hx}^{1E} = - K \left[(\psi_2 - \psi_1)\sin\theta_1 + \phi_1 - \phi_2 \cos(\theta_2 - \theta_1) \right]$$

$$g_{Hy}^{1E} = 0 \qquad\qquad (105)$$

$$g_{Hz}^{1E} = K \left[(\psi_2 - \psi_1)\cos\theta_1 - \phi_2 \sin(\theta_2 - \theta_1) \right].$$

We mention that the components in Equation (105) can also be obtained from the relation

$$\begin{pmatrix} g_{Hx}^{1E} \\ g_{Hy}^{1E} \\ g_{Hz}^{1E} \end{pmatrix} = -R^{12} \begin{pmatrix} g_{Hx}^{2E} \\ g_{Hy}^{2E} \\ g_{Hz}^{2E} \end{pmatrix}$$

where R^{12} is the transformation matrix connecting frame 2 to frame 1 and can be taken as:

$$R^{12} = \begin{pmatrix} \cos(\theta_2 - \theta_1) & 0 & \sin(\theta_2 - \theta_1) \\ 0 & 1 & 0 \\ -\sin(\theta_2 - \theta_1) & 0 & \cos(\theta_2 - \theta_1) \end{pmatrix}$$

Having determined the constraint torques the motion of the manipulator can be determined by using Equations (66), (68), (72), (101), (104) and (105) with n = 4 (n = 3 for Eq. (72)).

5.0 CONCLUSIONS

The principles of linear and angular momentum (Newton's and Euler's law) applied to each individual body in the chain provide a simple way of writing the equations of motion of a multi-body system. The unknown constraint forces and torques that appear in the equations of motion can be either eliminated or specified approximately using the constraint equations. It is shown that the elimination procedure leads to a densely coupled system of second-order equations which can be written in the vector-matrix form $B(\gamma)\,\ddot{\gamma} = f(\gamma,\dot{\gamma},t)$ where B is an $n' \times n'$ matrix and n' is the rotational degrees of freedom. On the other hand the second method of specifying approximately the unknown forces and torques leads to a system of 6n second-order equations where n is the number of rigid bodies in the system. The system of equations obtained by this method is simple and less densely coupled. Both these methods are general and can be easily implemented on the computer.

For studying the dynamics of multi-body systems several other methods, perhaps less general, have also been proposed. These may be found in References 9 - 14.

6.0 REFERENCES

1. Hooker, W.W.   *A Set of r Dynamical Attitude Equations for an Arbitrary n-Body Satellite Having r Rotational Degrees of Freedom.*
AIAA J., Vol. 8, 1970, pp. 1205-1207.

2. Wittenburg, J.   *Dynamics of Systems of Rigid Bodies.*
B.G. Teubner, Stuttgart, 1977.

3. Orlandea, N.
Chace, M.A.
Calahan, D.A.   *A Sparcity-Oriented Approach to the Dynamic Analysis and Design of Mechanical Systems — Part 1.*
Journal of Engineering for Industry, Trans. ASME, Series B., Vol. 99, 1977, pp. 773-779.

4. Mufti, I.H.   *Absolute Velocity and Acceleration in Moving Reference Frame.*
NRC, DME Laboratory Technical Report LTR-AN-3, National Research Council Canada, Ottawa, March 1974.

5. Greenwood, D.T.   *Principles of Dynamics.*
Prentice-Hall of Canada, Toronto, 1965.

6. Goldstein, H.   *Classical Mechanics.*
Addison-Wesley, Reading, Massachusetts, 1950.

7. Korn, G.
Korn, T.   *Electronic Analog and Hybrid Computers.*
McGraw-Hill, New York, 1964.

8. Mufti, I.H.
Goodanetz, J.W.
Gagne, R.E.   *Hybrid Computer Model of SRMS-End Effector Position Feedback.*
NRC, DME Laboratory Technical Report LTR-AN-31, National Research Council Canada, Ottawa, June 1978.

9. Uicker, J.J.   *Dynamic Behaviour of Spatial Linkages.*
Journal of Engineering for Industry, Trans. ASME, Series B, Vol. 91, 1969, pp. 251-265.

10. Vance, J.M.
Sitchin, A.   *Derivation of First-Order Difference Equations for Dynamical Systems By Direct Application of Hamilton's Principle.*
Journal of Applied Mech., Trans. ASME, Series E, Vol. 92, 1970, pp. 276-278.

11. Huston, R.L.
Passerello, C.   *On Multi-Rigid-Body System Dynamics.*
Journal Computers and Structures, Vol. 10, 1979, pp. 439-446.

12. Stepanenko, Y.
Vukobratovic, M.   *Dynamics of Articulated Open-Chain Active Mechanisms.*
Mathematical Biosciences, Vol. 28, 1976, pp. 137-170.

13. Vukobratovic, M.
Potkonjak, V.   *Contribution of the Forming of Computer Methods for Automatic Modelling of Spatial Mechanisms Motions.*
Mechanism and Machine Theory, Vol. 14, 1979, pp. 179-188.

14. Potkonjak, V.
Vukobratovic, V.   *Two New Methods for Computer Forming of Dynamic Equations of Active Mechanisms.*
Mechanism and Machine Theory, Vol. 14, 1979, pp. 189-200.